A METHOD OF AN EQUIVALENT EQUATION IN THE THEORY OF HEAT AND MASS TRANSFER

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The system of transfer equations in a two-phase medium reduces to a single effective equation which can be analyzed without significant difficulties (in comparison with the given system), with the accuracy of the problem formulation being practically preserved.

1. In the analysis of transfer processes in a heterogeneous medium the latter is often represented as a set of coexistent continua. Transfer is described separately for each continuum with account of heat and mass exchange between them [1-4]. A solution of the obtained system of equations even for the quasistationary representation of heat and mass exchange, as a rule, creates serious difficulties. In [5, 6] a system of transfer equations is proposed which reduces to a single equivalent equation, the analysis of which can be readily carried out.

We consider the following system of transfer equations:

$$\left(a\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\vartheta_1=\nu\Delta\vartheta_1+\vartheta_2-\vartheta_1,\quad \frac{\partial\vartheta_2}{\partial t}=\varepsilon\Delta\vartheta_2-\vartheta_2+\vartheta_1. \tag{1}$$

For the problem of heat exchange, ϑ_1 and ϑ_2 are dimensionless phase temperatures; ε is the ratio of the heat conductivities; $\mathbf{u}$ is the velocity of motion of the first phase, the second phase is at rest (for example, the heating of a granular mass by the flow of liquid). When $\mathbf{u} = 0$, the system (1) describes the filtering of the liquid in a so-called fissured and porous medium [3] (the medium with two systems of channels with significantly different permeabilities, the ratio of permeabilities $\varepsilon \ll 1$), ϑ_i, the liquid pressure in channels (for gas, the square of the pressures) The linearized equations of the filtering in media with double porosity with account of nonlinear elastic effects [7, 8] in the deformed fissured and porous medium [4] and the equations of the filtering of gas in the porous medium with account of sorption [9, 10] are similar in form to Eq. (1). The parameters $a > 0$, $\nu > 0$ are determined by the physical properties of the medium.

Below we consider the case $\varepsilon \ll 1$, which is a rather general one for the problems of transfer. The corresponding problems of filtration are presented above. In the problems of heat exchange in a dispersive medium the contact conductivity ($\varepsilon = 0$) is usually neglected. Situations are, however, possible when heat transfer by particles should be taken into account (for example, in vacuum treatment). Examples of solutions of (1) are known in the literature (see e.g., [11]).

In the present work, according to the method given in [5], system (1) reduces to a single effective equation. For the case when the characteristic time of the variation of the average characteristics considerably exceeds the characteristic time of the local variation of the process [in (1) $t \gg 1$], approximations to the solution are of a simple form convenient for practical applications. In actuality, the indicated case occurs, for example, for the quasistationary formulation of the problem (the selection of heat exchange between the phases in the form $\vartheta_2-\vartheta_1$, etc.).

2. Letting in (1)

$$\vartheta_i=\sum_{n=0}^{\infty}\varepsilon^n\vartheta_{in}, \tag{2}$$

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we obtain for the coefficients ϑ_{in} the equations

$$(a\partial/\partial t + \mathbf{u}\cdot\nabla)\vartheta_{1n} - \nu\Delta\vartheta_{1n} - \vartheta_{2n} + \vartheta_{1n} = 0,$$

$$\partial\vartheta_{2n}/\partial t + \vartheta_{2n} - \vartheta_{1n} = \Delta\vartheta_{2,n-1}\ (n = 0,\ 1,\ \ldots;\ \vartheta_{2,-1} = 0). \tag{3}$$

After substitution

$$\vartheta_{1n} = \varphi_{1n} - \Delta\vartheta_{2,n-1} \tag{4}$$

the second equation in (3) assumes the form

$$\partial\vartheta_{2n}/\partial t + \vartheta_{2n} - \varphi_{1n} = 0,$$

which leads to the formal expansion [5]:

$$\vartheta_{2n} = K\varphi_{1n},\ K \equiv \sum_{s=0}^{\infty}(-1)^s\,\partial^s/\partial t^s. \tag{5}$$

With account of (5), the first equation in (3) is written as follows:

$$L\vartheta_{1n} = K(\Delta\vartheta_{2,n-1})\quad (n = 0,\ 1,\ \ldots),$$

$$L \equiv L_1 - \sum_{s=2}^{\infty}(-1)^s\,\partial^s/\partial t^s,\quad L_1 \equiv (1 + a)\,\partial/\partial t + \mathbf{u}\cdot\nabla - \nu\Delta. \tag{6}$$

By multiplying the n-th equation (6) by ε^n and summing over n, with regard to (2), we obtain the equivalent equation

$$L\vartheta_1 = F(\vartheta_1),\ F(\vartheta_1) \equiv \sum_{n=1}^{\infty}\varepsilon^n\Delta^{(n)}(K^{(n+1)}\vartheta_1), \tag{7}$$

where the symbols $\Delta^{(n)}$ and $K^{(n+1)}$ designate n- and (n + 1)-fold application of the operators Δ and K, respectively. For $\varepsilon = 0$ Eq. (7) coincides with the analogous equation in [5].

We apply to (7) the Laplace transform in time. The operator $K^{(n)}$ can be shown to assume the form of $1/(1 + p)^n$, and Eq. (7) for the case of constant velocity **u** of the first phase assumes the form

$$\nu\Delta\bar{\vartheta}_1 - ab(p)\bar{\vartheta}_1 - \mathbf{u}\cdot\nabla\bar{\vartheta}_1 = -\sum_{n=1}^{\infty}\frac{\varepsilon^n}{(1 + p)^{n+1}}\Delta^{(n)}\bar{\vartheta}_1,$$

$$b(p) = p(p + d)/(1 + p),\ d = 1 + 1/a. \tag{8}$$

The following should be stressed. Despite the use of the formal expansion (5), Eq. (8) in final form is written for arbitrary values of the parameter of transformation p. Therefore, Eq. (8) coincides with the analogous equation obtained directly from (1) after eliminating ϑ_2 and applying the Laplace transform. Therefore, for the initial problem under consideration, solution (8) is exact.

The determination of the inverse transform for the solution of Eq. (8) in the general case causes considerable difficulties. In correspondence with the approach [5], different approximations of the process can be constructed, retaining in (7) derivatives of different orders with respect to time. The zeroth approximation corresponds to the stationary case. The first approximation is described by an equation of parabolic type:

$$L_1\vartheta_1 = F(\vartheta_1),\quad K \equiv 1 - \partial/\partial t, \tag{9}$$

the second approximation is described by an equation of elliptic type*:

$$(L_1 - \partial^2/\partial t^2)\vartheta_1 = F(\vartheta_1),\quad K \equiv 1 - \partial/\partial t + \partial^2/\partial t^2. \tag{10}$$

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*Under certain assumptions heat transfer in a dispersive layer can be described by a hyperbolic equation [12].

The procedure for retaining successively the derivatives with respect to time of different orders (beginning from the first one) in (7) corresponds to the expansion of the solution of Eq. (8) in a series in the parameter of transformation p, which is possible under the condition that $t \gg 1$ ($p \ll 1$). As is pointed out above, this condition holds actually for the quasistationary definition of the problem.

By applying to (9) and (10) the Laplace transform in time and carrying out the inversion, one can find the approximations mentioned above to the solution with an arbitrary degree of precision with respect to ε. The suitability of the approximations built up is determined by checking them against the exact solution, the inverse transform of the solution of Eq. (8).

Further we assume that **u** = 0 (this corresponds, for example, to the cases of filtration mentioned above) and limit ourselves to one-dimensional processes.

3. For the boundary conditions $\vartheta_1 = \vartheta_2 = \vartheta_*$ with x = 0 and zero initial conditions (for example, the problem on starting a gallery in a semiinfinite region and heating the mass by the flow of liquid), solution (8) with accuracy to ε^2 is of the form

$$p\bar{\vartheta}_1 = \vartheta_* [1 + \varepsilon\beta(p)x/(2\nu(1+p)^2) + \varepsilon^2(\ldots) + \ldots] \exp(-\beta(p)x), \quad \beta(p) = \sqrt{ab(p)/\nu}. \tag{11}$$

The application of the Laplace transform in time [13] to (9) and (10) does not cause difficulties. We list the final results:

the solution of Eq. (9)

$$\frac{\vartheta_1}{\vartheta_*} = \operatorname{erfc}\xi + \varepsilon\frac{\xi\exp(-\xi^2)}{\sqrt{\pi\nu}}\left(1 - \frac{2\xi^2-1}{t}\right), \quad \xi = \frac{x}{2}\sqrt{\frac{1+a}{\nu t}}, \tag{12}$$

the solution of Eq. (10)

$$\frac{\vartheta_1}{\vartheta_*} = \operatorname{erfc}\xi + \frac{\xi\exp(-\xi^2)}{\sqrt{\pi}}\left[\frac{2\xi^2-1}{2(1+a)t} + \frac{\varepsilon}{\nu}\left(1 - \frac{5+4a}{4(1+a)t}(2\xi^2-1)\right)\right], \tag{13}$$

where $\operatorname{erfc}\xi = 1 - \operatorname{erf}\xi$ is an error function.

For an arbitrary boundary condition $\vartheta_1(0, t) = f(t)$, the solution is defined by the Duhamel integral

$$\int_0^t \vartheta_1(x, t-\omega)\frac{df(\omega)}{d\omega}\,d\omega. \tag{14}$$

The function ϑ_2 is determined from Eqs. (2), (5), (4).

When ε = 0, as is shown in [5], functions (12), (13) represent odd and even with respect to x approximations of the solution of system (1) [or of the inverse transform (11)]. It can be determined in a similar way that the terms of the order of ε are approximated evenly.

In correspondence with (12), (13), we write down for the process of filtration the approximation of the yield (for the problem of heat exchange, the approximation of heat flow) $q = (\partial\vartheta_1/\partial x + \varepsilon\partial\vartheta_2/\partial x)_{x=0}$:

parabolic approximation

$$q = -\frac{\vartheta_*}{2}\sqrt{\frac{1+a}{\pi\nu t}}\left[2 + \varepsilon\left(2 - \frac{1}{\nu} + \frac{1}{t} - \frac{1}{\nu t}\right)\right], \tag{15}$$

elliptic approximation

$$q = -\frac{\vartheta_*}{2}\sqrt{\frac{1+a}{\pi\nu t}}\left[2+\frac{1}{2(1+a)t}+\varepsilon\left(2-\frac{1}{\nu}+\frac{1}{4(1+a)t}\left(5+4a+\frac{6}{t^2}+\frac{45}{2t^3}\right)\right)\right]. \quad (16)$$

For practical purposes when t ≫ 1 it suffices to use, according to (15), (16), the expression for the yield

$$q \approx -\vartheta_* \sqrt{\frac{1+a}{\pi\nu t}}. \quad (17)$$

According to (12)-(17), when t ≫ 1 the filtration in the fissured and porous medium or filtration in the porous medium with consideration for sorption occurs in the way similar to that in the porous medium (without sorption) with an effective coefficient of piezoconductivity $\varkappa = \nu/(1+a)$ (with the exponential approach to the self-similarity regime), which corresponds to theoretical and experimental results [14-16]. Therefore, the equivalent equation describes qualitatively correctly the given cases of transfer and allows one to evaluate readily the effective parameters of the system, which is of interest by itself. We remark that the effective piezoconductivity of the system is always less than the piezoconductivity of fissures (according to the chosen designations the latter is equal to ν) and depends on the parameter a.

As an example, we estimate the parameter a for filtering in the elastically compressible fissured and porous medium. According to [4]:

$$a = m_1^\circ [m_2^\circ (p^\circ - \sigma)(\beta_\rho + \beta_m)]^{-1},$$

where m_1° and m_2° are porosities of fissures and blocks at the initial formational pressure p°; β_ρ and β_m, coefficients of compressibility of the liquid and pores of blocks; σ, the critical pressure (when $p_1 = \sigma$ the fissures are closed). In actuality, $p^\circ - \sigma \sim (10^5\text{-}10^7)$ N/m². Assuming for the parameters usual values [17] $\beta_\rho \sim (10^{-8}\text{-}10^{-9})$ m²/N, $\beta_m \sim 10^{-9}$ m²/N, $m_1^\circ/m_2^\circ \sim 10^{-j}$ (j = 1, 2, ...), we determine $a \sim (10^{1-j}\text{-}10^{3-j})$. For the model [3] of the fissured and porous medium $a \ll 1$ due to the assumption that fissures are weakly compressible.

In order to estimate the accuracy of the approximations constructed (12), (13) we compare for ε = 0 the corresponding expressions for the yield (15), (16) with the exact expression determined from (11):

$$q = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp(pt)\left(\frac{\partial\bar{\vartheta}_1}{\partial x}\right)_{x=0} dp = -\frac{\vartheta_*}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp(pt)\frac{\beta(p)}{p}\,dp. \quad (18)$$

Inside of the selected contour of integration (Fig. 1) the integrand in (18) is single-valued and analytic (p = 0, −1, −d, ∞ are the branching points); therefore,

$$q = -\frac{\vartheta_*}{2\pi i}\left(\int_{C_R} + \int_{C_r^{(1)}} + \int_{C_r^{(2)}} + \int_{C_r^{(3)}} + \int_{\mathrm{I}} + \int_{\mathrm{II}}\right), \quad (19)$$

where I and II are the horizontal edges of the cut (the direction of pass-around of the contour is shown by an arrow). All the values in Figs. 1-3 are dimensionless.

By evaluating circular integrals in a regular way [18], one can demonstrate that

$$\lim_{R\to\infty}\int_{C_R} = \lim_{r\to 0}\int_{C_r^{(i)}} = 0 \qquad (i = 1,\ 2,\ 3).$$

Taking into account that on the lower edge of the cut p = x exp (−πi), on the upper cut, p = x exp (πi), one can find that

$$\int_{\mathrm{I}} + \int_{\mathrm{II}} = 2i\int_r^{1-r} G(x,\ t)\,dx + 2i\int_{d+r}^{R} G(x,\ t)\,dx,$$

$$G(x,\ t) = \sqrt{\frac{x-d}{x(x-1)}}\exp(-xt).$$

By taking the limit in (19) with $r \to 0$, $R \to \infty$, we finally obtain

$$q = -\frac{\vartheta_*}{\pi}\sqrt{\frac{a}{\nu}}\left(\int_0^1 G(x, t)\,dx + \int_d^\infty G(x, t)\,dx\right). \tag{20}$$

In the general case, dependence (20) $q = q(t)$ is calculated numerically. When $a \gg 1$ we obtain $d \approx 1$ and

$$q \approx -\frac{\vartheta_*}{\pi}\sqrt{\frac{a}{\nu}}\int_0^\infty \frac{\exp(-xt)}{\sqrt{x}}\,dx = -\vartheta_*\sqrt{\frac{a}{\pi\nu t}},$$

which coincides with approximations (15), (16) (Fig. 2).

4. In the problem on starting a bore with a constant yield q we neglect the flow on blocks with the purpose of simplifcation [in (1) $\varepsilon = 0$]. The Laplace transform of the solution of the equivalent equation (7) is of the form (it coincides with the corresponding expression [14] for the porous medium):

$$\bar{\vartheta}_1 = -\frac{q}{pr_0\beta(p)}K_0(\beta(p)r)/K_1(\beta(p)r_0),$$

where K_0 and K_1 are the modified Bessel functions; r_0, a bore radius; the expression $\beta(p)$ is given in (11). By using the condition $\beta(p)r_0 \ll 1$ and making substitution

$$K_1(\beta(p)r_0) \approx 1/(\beta(p)r_0),$$

we obtain

$$p\bar{\vartheta}_1 = -qK_0(\beta(p)r). \tag{21}$$

By expanding the right-hand part in (21) in a series of powers $p \ll 1$, we determine the analog of Eqs. (15), (16):

$$\frac{p\bar{\vartheta}_1}{q} = -K_0(\sqrt{\delta p}) - \frac{p\sqrt{\delta p}}{2(1+a)}K_1(\sqrt{\delta p}) + \cdots, \quad \delta = r\sqrt{\frac{1+a}{\nu}}.$$

The inverse function is determined from the expression:

$$\frac{2\vartheta_1}{q} = \mathrm{Ei}(-\xi^2) - \frac{\xi^2}{\sqrt{(1+a)t}}\exp(-\xi^2) + \cdots \approx \mathrm{Ei}(-\xi^2),$$
$$\xi = \frac{r}{2}\sqrt{\frac{1+a}{\nu t}}, \tag{22}$$

where $\mathrm{Ei}(-\xi^2)$ is an integral exponential function. According to (22), a bore pressure changes in time according to the law:

$$-\frac{2\vartheta_1(r_0, t)}{q} = \ln\frac{2.25t}{(1+a)r_0^2} + \ln\nu. \tag{23}$$

For an arbitrary boundary condition $q = f(t)$ the solution is written similarly to (14).

The efficacy of approximation (22) can be estimated by comparing variation in the bore pressure (23) with a similar relationship (exact), determined from (21):

$$\frac{\vartheta_1(r_0, t)}{q} = C + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\exp(pt)}{p}\ln\left(\beta(p)\frac{r_0}{2}\right)dp. \tag{24}$$

When writing (24) we used an asymptotic approximation $K_0(z) \approx -C - \ln(z/2)$ for $z \ll 1$, where $C = 0.5772$ is Euler's constant; the integration contour is shown in Fig. 1. When $a \gg 1$ dependences (23) and (24) coincide.

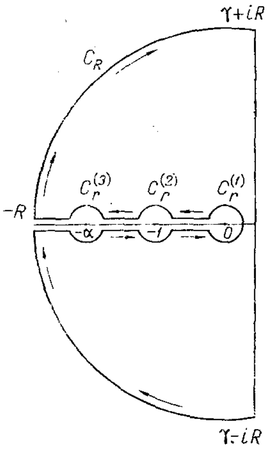

Fig. 1

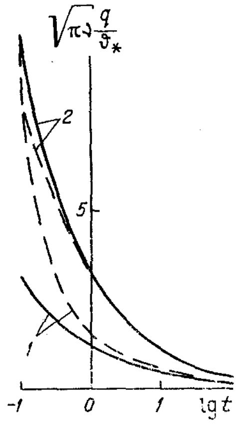

Fig. 2

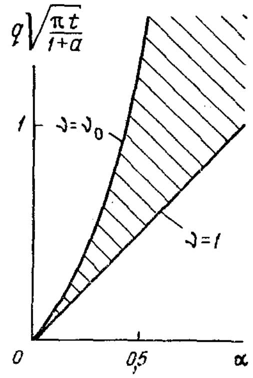

Fig. 3

Fig. 1. Integration countour for calculating integrals (18), (24).

Fig. 2. Dependence of the gallery yield on time: a = 0.1 (1), 10 (2); solid lines, parabolic approximation; dashed lines, elliptical approximation.

Fig. 3. Dependence of the yield on depression.

5. We show how to apply the equivalent equation to nonlinear problems on the example of the filtration in the elastic-compressible fissured and porous medium [4]. For this case the parameter ν in (10) characterizes the variable (due to compressibility) permeability of fissures:

$$(1-\alpha)^3=\nu_0\leqslant\nu=(1+\vartheta_1)^3\leqslant 1,$$

where $\alpha = (p^0 - p_0)/(p^0 - \sigma)$, $0 < \alpha < 1$ is a dimensionless depression* (p_0 is a dimensional face pressure).

Letting in (16) $\nu_0 \leq \nu \leq 1$, one can estimate the interval of variation of the gallery debit for a nonlinear system (dashed region in Fig. 3) and, by using experimental data, draw a conclusion on expediency of the linearization of the problem. It is known from the experimental data [16] and the analysis of a stationary case [19] that, when the depression increases, the debit increases and reaches a saturation state by the moment when fissures close, any further increase in debit is possible at the expense of blocks with small permeability only. On the basis of that one can conclude the following.

For sufficiently small values of depression (while fissures are weakly compressible) the dashed region in Fig. 3 is narrow; therefore, the debit of the nonlinear system can be assumed to be equal to the debit of the porous medium with the permeability of fissures (straight line $\nu = 1$). For example, when $\alpha \leq 0.125$ an error in determining the debit does not exceed 20%. When the depression increases, the interval of variation in debit increases sharply; therefore, the error in evaluating the debit on the basis of a linearized system can appear to be intolerably large. An analogous conclusion can be drawn with regard to the determination of the heat flow in the problem of heat exchange.

The same conclusion follows on the basis of (22), (23) for the problem on starting a bore with a constant debit. For small depressions ($\nu \approx 1$) the curves of the pressure recovery can be interpreted as for the porous medium. When ν decreases, the error in determining parameters of a collector becomes uncontrolled.

NOTATION

ϑ, t, x, u, dimensionless temperature (pressure), time, coordinate, velocity; ϑ, the value of x = 0; a, ε, ν, dimensionless parameters in (1); d, parameter in (8); ξ, self-similarity variable; q, debit (heat flux); $\varkappa$, piezoconductivity; m°, porosity at the initial pressure p°; β_p, β_m, compressibility factors of liquid and block pores; σ, critical pressure; α, dimensionless depression; p, parameter of the Laplace transform; b(p), function in (8);

*The case $\alpha = 1$ ($p^0 = 0$ and the fissures are joined in a zone near the face) is ignored, since $\varepsilon \ll 1$.

$\beta(p)$, in (11), r_0, dimensionless radius of a bore; K, operator in (5); L, L_1, operators in (6); $F(\vartheta_1)$, function in (7); G(x, t), function in (20); ∇, gradient; Δ, the Laplace operator. Indices: 1, 2, refer to fissures and blocks; a superscript bar, the Laplace transform.

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